

Deconstructing non-dissipative non-Dirac-hermitian relativistic quantum systems

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Abstract

A method to construct non-dissipative non-Dirac-hermitian relativistic quantum system that is isospectral with a Dirac-hermitian Hamiltonian is presented. The general technique involves a realization of the basic canonical (anti-)commutation relations involving the Dirac matrices and the bosonic degrees of freedom in terms of non-Dirac-hermitian operators, which are hermitian in a Hilbert space that is endowed with a pre-determined positive-definite metric. Several examples of exactly solvable non-dissipative non-Dirac-hermitian relativistic quantum systems are presented by establishing/employing a connection between Dirac equation and supersymmetry.

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1 Introduction

An emergent view in the current literature is that a formulation of non-dissipative quantum systems is possible without the requirement of the Dirac-hermiticity[1, 2, 3]. The quantum systems constructed in this way are indeed hermitian in a Hilbert space that is endowed with a positive-definite metric different from the identity operator and hence, a modified inner-product. The construction of the metric in the Hilbert space is a non-trivial task and is not known for a majority of the examples considered in this context. A lack of knowledge of the metric makes even an otherwise exactly solvable non-Dirac-hermitian quantum system incomplete, since no expectation values of physical variables can be calculated explicitly. An approach taken in Refs. [4, 5] in this regard was to construct non-dissipative non-Dirac-hermitian quantum systems with a pre-determined positive-definite metric in the Hilbert space. The quantum systems constructed following this approach may serve as prototype examples of non-dissipative non-Dirac-hermitian quantum models with a complete description. These systems may be used to test different ideas related to the subject[4, 5, 6], including validity of any approximate and/or numerical method.

Recently, non-Dirac-hermitian relativistic quantum systems with entirely real spectra have been studied[7, 8, 9, 10, 11, 12, 13]. An optical realization of relativistic quantum system with non-Dirac-hermitian interaction has also been proposed[13]. This proposal assumes a significance within the background that ideas emanating from combined parity and time-reversal symmetric optical systems have been verified experimentally[14, 15]. A majority of the non-Dirac-hermitian relativistic models proposed so far are incomplete in the sense that the positive-definite metric and/or the modified inner-product in the Hilbert space is not known.

The purpose of this article is to present a class of non-dissipative non-Dirac-hermitian relativistic quantum systems admitting entirely real spectra and unitary time evolution. The metric and hence, the inner-product in the Hilbert space is specified explicitly so as to facilitate the computations of expectation values of different physical observables. The general method is an extension of the technique employed in Refs. [4, 5] to relativistic systems. In particular, a non-Dirac-hermitian realization of Dirac matrices and the bosonic variables is used, which are hermitian in a Hilbert space that is endowed with a pre-determined positive-definite metric. Apart from the general results, several examples of exactly solvable non-dissipative non-Dirac-hermitian quantum systems are presented by employing a connection between supersymmetry and Dirac equation.

2 Preliminaries

The Pauli matrices σ^a , position variables x^a and the momentum operators p^a are taken to be hermitian in the Hilbert space \mathcal{H}_D with the standard inner-product $\langle \cdot | \cdot \rangle$. A Hilbert-space \mathcal{H}_{η_+} that is endowed with the positive-definite

metric η_+ ,

$$\eta_+ := \exp\left(-2\vec{J} \cdot \hat{n}\phi\right), \quad \vec{J} := \vec{L} + \frac{\vec{\sigma}}{2}, \quad \hat{n} \cdot \hat{n} = 1, \quad (\phi, n^a) \in R, \quad a = 1, 2, 3, \quad (1)$$

and the inner-product $\langle\langle \cdot | \cdot \rangle\rangle_{\mathcal{H}_{\eta_+}} := \langle \cdot | \eta_+ \cdot \rangle$ is introduced. The operator \vec{L} with $L^a = \epsilon^{abc} x^b p^c$ denotes orbital angular momentum in \mathcal{H}_D , where ϵ^{abc} denotes the three dimensional Levi-Civita tensor. The operator \vec{J} can thus be identified as the total angular momentum of the system. It may be noted that the metric η_+ can be decomposed as a direct product of a purely bosonic and purely fermionic metrics. In particular, $\eta_+ = \eta_b \otimes \eta_f$ where $\eta_b := \exp(-2\vec{L} \cdot \hat{n}\phi)$ and $\eta_f := \exp(-\vec{\sigma} \cdot \hat{n}\phi)$. The positivity of the metric follows from the fact that the eigen values of the operator $\vec{J} \cdot \hat{n}$ are real. With the introduction of the similarity operator ρ ,

$$\rho := \sqrt{\eta_+} = \exp\left(-\vec{J} \cdot \hat{n}\phi\right), \quad (2)$$

a set of non-Dirac-hermitian matrices Σ^a , position variables X^a and the momentum operators P^a may be introduced as follows[2, 4, 5]:

$$\begin{aligned} \Sigma^a &:= (\rho)^{-1} \sigma^a \rho = \sum_{b=1}^3 R^{ab} \sigma^b, \\ X^a &:= (\rho)^{-1} x^a \rho = \sum_{b=1}^3 R^{ab} x^b, \\ P^a &:= (\rho)^{-1} p^a \rho = \sum_{b=1}^3 R^{ab} P^b, \\ R^{ab} &\equiv n^a n^b (1 - \cosh\phi) + \delta^{ab} \cosh\phi + i\epsilon^{abc} n^c \sinh\phi. \end{aligned} \quad (3)$$

The non-Dirac-hermiticity of Σ^a , X^a and P^a follows from the fact that $(R^{ab})^* \neq R^{ab}$, where f^* denotes the complex conjugate of f . It can be shown by using the identity $\sum_{a=1}^3 R^{ab} R^{ac} = \delta^{bc}$ that $\vec{\Sigma} \cdot \vec{P} = \vec{\sigma} \cdot \vec{p}$. This implies that $\vec{\sigma} \cdot \vec{p}$ and $\vec{\Sigma} \cdot \vec{P}$ are hermitian both in \mathcal{H}_D as well as in \mathcal{H}_{η_+} . The matrices Σ^a obey the same algebra satisfied by the Pauli matrices:

$$\begin{aligned} [\Sigma^a, \Sigma^b] &= 2i\epsilon^{abc} \Sigma^c, \quad \{\Sigma^a, \Sigma^b\} = 2\delta^{ab} \\ \Sigma_{\pm} &:= \frac{1}{2} (\Sigma^1 \pm i\Sigma^2), \quad \{\Sigma_-, \Sigma_+\} = 1, \quad \Sigma_{\pm}^2 = 0, \end{aligned} \quad (4)$$

and are hermitian in the vector space \mathcal{H}_{η_+} . Similarly, the position and the momentum operators satisfy the basic canonical commutation relations,

$$[X^a, X^b] = 0 = [P^a, P^b], \quad [X^a, P^b] = i\delta^{ab}, \quad (5)$$

and are hermitian in \mathcal{H}_{η_+} . All other commutators involving Σ^a , X^a and P^a vanish identically.

3 Dirac Equation in 1 + 1 dimensions

In this section, a convenient notation $x^1 \equiv x$ and $p^1 \equiv p_x$ will be used to denote the position and the momentum operators. A non-Dirac-hermitian relativistic Hamiltonian in 1 + 1 dimensions is introduced as follows,

$$\begin{aligned} H_{1D} &= \sigma^2 p_x + \sigma^1 A(x) + \sigma^3 B(x) + V(x) \\ A(x) &\equiv M(x) \cosh \phi + iP(x) \sinh \phi, \\ B(x) &\equiv iM(x) \sinh \phi - P(x) \cosh \phi, \phi \in R, \end{aligned} \quad (6)$$

where $M(x)$, $P(x)$ and $V(x)$ are real functions of the coordinate. The complex functions $A(x)$ and $B(x)$ can be interpreted as scalar and pseudo-scalar potentials, respectively. The potential $V(x)$ is the time-component of a Lorentz two-vector potential. The space-component of the same two-vector has been taken as zero, since it can always be gauged away in 1 + 1 dimensions. The Lorentz covariance of the Dirac equation can be shown by introducing $\gamma^0 := \sigma^1$, $\gamma^1 := i\sigma^3$. The matrix γ_5 may be defined as $\gamma^5 = i\sigma^2$. Various representations of the Dirac equation in 1 + 1 dimensions have been used previously [8, 10, 13, 16]. All these representations are related to each other through unitary transformations. For example, the unitary operator $U = \exp(-\frac{i\pi}{4}\sigma^1)$ relates the representation used in this article to the one used in Ref. [13] in the context of optical realization of non-Dirac-hermitian relativistic quantum systems.

The Hamiltonian H_{1D} is non-Dirac-hermitian. With the introduction of the hermitian and positive-definite operators,

$$\rho_{1D} := e^{-\frac{\phi}{2}\sigma^2}, \quad \eta_{1D} := \rho_{1D}^2 = e^{-\phi\sigma^2}, \quad (7)$$

the non-Dirac-hermitian Hamiltonian H_{1D} can be mapped to a Dirac-hermitian Hamiltonian,

$$\begin{aligned} h_{1D} &= \rho_{1D} H_{1D} \rho_{1D}^{-1} \\ &= \sigma^2 p_x + \sigma^1 M(x) - \sigma^3 P(x) + V(x). \end{aligned} \quad (8)$$

The similarity operator ρ_{1D} is obtained from Eq. (2) by choosing $\vec{L} = 0$, $\hat{n}^1 = 0 = \hat{n}^3$, $\hat{n}^2 = 1$. The Hamiltonian H_{1D} is hermitian in the Hilbert space $\mathcal{H}_{\eta_{1D}}$ that is endowed with the metric η_{1D} and a modified norm $\langle\langle \cdot | \cdot \rangle\rangle_{\eta_{1D}} = \langle \cdot | \eta_{1D} \cdot \rangle$. The Hamiltonian H_{1D} and h_{1D} are isospectral. The function $M(x)$ and $P(x)$ may be identified as the scalar and the pseudo-scalar interactions, respectively, for the Dirac-hermitian Hamiltonian h_{1D} .

3.1 Case I: Pure scalar interaction

The eigen value equation for h_{1D} can be solved exactly for specific choices of the interactions. The first example considered here is the vanishing vector potential and vanishing $P(x)$. The Hamiltonian h_{1D} for this case may be identified as

the supercharge of a supersymmetric quantum system with $H_s := h_{1D}^2$ as the corresponding supersymmetric Hamiltonian[16]. In particular,

$$H_s = p_x^2 + M^2 - \sigma^3 M', \quad (9)$$

where $f'(x)$ denotes the derivative of $f(x)$ with respect to its argument. An exhaustive list of M for which H_s is exactly solvable is known[16]. The non-Dirac hermitian Hamiltonian H_{1D} ,

$$H_{1D} = \sigma^2 p_x + \sigma^1 M(x) \cosh \phi + i \sigma^3 M(x) \sinh \phi, \quad (10)$$

is thus exactly solvable corresponding to each M for which H_s is exactly solvable. The energy eigen values E of H_{1D} is determined as $E = \pm \sqrt{E_s}$, where E_s is the energy of H_s . It may be recalled here that E_s being the energy of a supersymmetric theory, it is always semi-positive definite. The eigen-spinor Ψ of H_{1D} is related to the eigen functions ψ of H_s as, $\Psi = \rho_{1D}^{-1} \psi$. A complete set of orthonormal eigen functions ψ in \mathcal{H}_D ensures orthonormality of Ψ in $\mathcal{H}_{\eta_{1D}}$.

3.2 Case II: Scalar & Pseudo-scalar interactions

The second example describes the case of a vanishing vector potential and non-vanishing $M(x)$ and $P(x)$. The square of the operator h_{1D} for this limiting case reads,

$$h_{1D}^2 = p_x^2 + M^2 + P^2 - \sigma^3 M' - \sigma^1 P'. \quad (11)$$

A unitary transformation $U := \exp(-i \frac{\theta}{2} \sigma^2)$, $\theta \in [0, 2\pi]$ maps h_{1D}^2 to the Hamiltonian H_1 ,

$$\begin{aligned} H_1 &:= U^{-1} h_{1D}^2 U \\ &= p_x^2 + M^2 + P^2 + \sigma^1 (M' \sin \theta - P' \cos \theta) \\ &\quad - \sigma^3 (P' \sin \theta + M' \cos \theta). \end{aligned} \quad (12)$$

It may be noted that U is unitary in \mathcal{H}_D as well as in $\mathcal{H}_{\eta_{1D}}$. The term containing σ^1 vanishes identically for a fixed value of θ , when the following relation involving the scalar and the pseudo-scalar potential is satisfied:

$$P = a_0 M + b_0, \quad a_0, b_0 \in \mathbb{R}, \quad \theta = \tan^{-1} a_0. \quad (13)$$

Thus, the Hamiltonian H_1 becomes diagonal in this limit and depends on only one arbitrary function $M(x)$.

The Hamiltonian H_1 can be re-written in a manifestly supersymmetric form,

$$\begin{aligned} \tilde{H} &:= H_1 - \frac{b_0^2}{1 + a_0^2} \\ &= p_x^2 + W^2 - \sigma_3 W', \end{aligned} \quad (14)$$

where the superpotential $W(x)$ is introduced as,

$$W(x) \equiv \sqrt{1 + a_0^2} M + \frac{a_0 b_0}{\sqrt{1 + a_0^2}}. \quad (15)$$

Thus, the known relation between supersymmetry and the Dirac equation in \mathcal{H}_D [16, 17] is extended to the case of non-Dirac-hermitian quantum systems. There is an exhaustive list of superpotentials $W(x)$ for which \tilde{H} is exactly solvable[16]. The non-Dirac-hermitian relativistic Hamiltonian,

$$H_{1D} = \sigma^2 p_x + \sigma^1 [e^\phi M_+ + e^{-\phi} M_-] + \sigma^3 [e^{(\phi - i\frac{\pi}{2})} M_+ + e^{-(\phi - i\frac{\pi}{2})} M_-] \\ M_\pm \equiv \frac{1 \pm ia_0}{\sqrt{1 + a_0^2}} W(x) - \frac{b_0}{1 + a_0^2} (a_0 \mp i), \quad (16)$$

is exactly solvable for each superpotential $W(x)$ for which \tilde{H} is exactly solvable. Moreover, a consistent quantum description of H_{1D} , including entirely real spectra and unitary time-evolution is admissible. An eigen-spinor $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ of H_{1D} with energy eigenvalues E is related to the corresponding eigen-functions $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ of \tilde{H} with energy eigenvalues \tilde{E}_{1D} through the relation,

$$\Psi = (\rho_{1D}^{-1} U) \psi. \quad (17)$$

The orthonormality of Ψ in $\mathcal{H}_{\eta_{1D}}$ is guaranteed, if the eigenfunctions ψ are constructed as a complete set of orthonormal eigenfunctions in \mathcal{H}_D . The eigenvalues E are determined in terms of the eigenvalues \tilde{E} through the relation,

$$E = \pm \sqrt{\tilde{E} + \frac{b_0^2}{1 + a_0^2}}. \quad (18)$$

The parameter ' b_0 ' can be chosen as zero without any loss of generality, since \tilde{E} being the energy eigen values of a supersymmetric theory is always semi-positive definite. In this limit, $E = \pm \sqrt{\tilde{E}}$ and the zero-energy ground state of the supersymmetric Hamiltonian \tilde{H} corresponds to the zero modes of the Dirac Hamiltonian H_{1D} given in Eq. (16). The choice of a reflection-less superpotential $W(x)$ produces a perfectly transparent potential for the non-Dirac-hermitian Hamiltonian.

3.3 Case III: Scalar, Pseudo-scalar & Vector interactions

The third example presented in this article is for non-vanishing $M(x)$, $P(x)$ and vector potential $V(x)$. The eigen-value equation of h_{1D} with arbitrary $P(x)$, $M(x)$ and $V(x)$ reads,

$$\begin{pmatrix} P(x) + V(x) & -ip_x + M(x) \\ ip_x + M(x) & -P(x) + V(x) \end{pmatrix} \begin{pmatrix} \psi^{(+)} \\ \psi^{(-)} \end{pmatrix} = E \begin{pmatrix} \psi^{(+)} \\ \psi^{(-)} \end{pmatrix}, \quad (19)$$

which can be decoupled by introducing the functions $\theta_\pm(x)$, $\chi_\pm(x)$,

$$\chi_\pm(x) \equiv \frac{\theta'_\pm(x)}{2\theta_\pm(x)}, \quad \theta_\pm(x) \equiv E \pm P(x) - V(x), \quad (20)$$

and redefining $\psi^{(\pm)}$ in terms of $\tilde{\psi}^{(\pm)}$ as,

$$\tilde{\psi}^{(\pm)}(x) \equiv \theta_{\pm}^{-\frac{1}{2}} \psi^{(\pm)}(x). \quad (21)$$

The decoupled equations have the form of stationary state Schrödinger equations,

$$\begin{aligned} H_{\pm} \tilde{\psi}^{(\pm)} &= E^2 \tilde{\psi}^{(\pm)}, \quad H_{\pm} := -\frac{d^2}{dx^2} + V_{\pm}(x), \\ V_{\pm} &\equiv M^2 + P^2 - V^2 + 2EV + \chi_{\pm}^2 - \chi'_{\pm} \mp (M' - M\chi_{\pm}). \end{aligned} \quad (22)$$

Unlike the previous two cases, the potentials V_{\pm} appearing in the effective Schrödinger equations explicitly depend on the energy E of the Dirac equation. In general, the potentials V_{\pm} can not be identified as partner superpotentials of a supersymmetric Hamiltonian. Such a scenario may arise for very specific choices of the interactions, the study of which is beyond the scope of this article. Nevertheless, either H_+ or H_- can be embedded in a supersymmetric theory and the eigen-value equation can be solved analytically.

The eigen value equation of the Hamiltonian h_{1D} can be solved by first identifying an exactly solvable H_- or H_+ . The eigenfunction $\tilde{\psi}^{(+)}$ of H_+ can then be determined in terms of the eigenfunction $\tilde{\psi}^{(-)}$ of H_- or the vice versa, by using the equations (19), (20), and (21). A choice of the vector potential in terms of $P(x)$ as,

$$V(x) = P(x) + a_1, \quad a_1 \neq E, \quad a_1 \in R, \quad (23)$$

leads to vanishing χ_+ . With a further choice of $P(x) = b_1 M(x)$, $b_1 \in R$, the potential V_{\pm} reads,

$$\begin{aligned} V_+ &= W^2 - W' + a_1(2E - a_1) - b_1^2(a_1 - E)^2, \\ V_- &= W^2 + W' + a_1(2 - a_1) - b_1^2(a_1 - E)^2 + [\chi_-^2 - \chi'_- - M\chi_-], \\ W &\equiv M - b_1(a_1 - E), \quad \chi_- = -\frac{2b_1 W'(x)}{(1 + 2b_1^2)(E - a_1) - 2b_1 W}. \end{aligned} \quad (24)$$

Although H_+ and H_- do not become super-partners of each other for this case, H_+ can be embedded in a supersymmetric theory. A supersymmetric quantum system in \mathcal{H}_D may be introduced in terms of the operators A , A^\dagger and the Hamiltonian $H^{(1)}, H^{(2)}$ as follows:

$$\begin{aligned} A &= ip_x + W(x), \quad A^\dagger = -ip_x + W(x) \\ H^{(1)} &:= A^\dagger A = -\frac{d^2}{dx^2} + W^2 - W', \\ H^{(2)} &:= AA^\dagger = -\frac{d^2}{dx^2} + W^2 + W'. \end{aligned} \quad (25)$$

The eigenfunctions and the corresponding eigenvalues of $H^{(1)}$ and $H^{(2)}$ are denoted as $\psi_n^{(1)}, e_n^{(1)}$ and $\psi_n^{(2)}, e_n^{(2)}$, respectively. The Hamiltonian H_+ and $H^{(1)}$

differ by a constant and commute with each other. Thus, the eigen-values E_n^\pm and the corresponding eigen functions $\psi_n^{(+)}$ of h_{1D} are determined as,

$$E_n^\pm = a_1 \pm \left(\frac{e_n^{(1)}}{1 + b_1^2} \right)^{\frac{1}{2}}, \quad \psi_n^{(+)} = \sqrt{E_n^\pm - a_1} \psi_n^{(1)}. \quad (26)$$

The convention to be followed henceforth is that for positive(negative) energy solutions $E^+(E^-)$ will be taken in the expressions of ψ_n^+ and ψ_n^- . It follows from Eqs. (19), (20), (21), (25) and (26) that the eigen functions $\psi_n^{(-)}$ of h_{1D} corresponding to energy eigen values E_n^\pm is determined in terms of $\psi_n^{(1)}$ as,

$$\begin{aligned} \psi_n^{(-)} &= \frac{1}{\sqrt{E_n^\pm - a_1}} [A - b_1(E_n^\pm - a_1)] \psi_n^{(1)} \\ &= \frac{1}{\sqrt{E_n^\pm - a_1}} \left[\sqrt{e_n^{(1)}} \psi_{n-1}^{(2)} - b_1(E_n^\pm - a_1) \psi_n^{(1)} \right], \end{aligned} \quad (27)$$

where $\psi_0^{(2)} = 0 = \psi_{-1}^{(2)}$. The Hamiltonian $H^{(1)}$ is exactly solvable for several choices of $W(x)$ corresponding to shape invariant potentials[16]. Thus, the Hamiltonian h_{1D} and hence, H_{1D} are also exactly solvable for the specific forms of $V(x)$, $P(x)$ and $M(x)$ mentioned above for which $W(x)$ produces shape-invariant potentials.

A few comments are in order before the end of this section:

- (i) A choice $V(x) = -P(x) + a_2$, $a_2 \in R$ leads to vanishing χ_- . With a further choice of $P(x) = b_2 M(x)$, $b_2 \in R$ and following the procedure outlined above, the complete eigen value equation can be solved analytically by embedding H_- in a supersymmetric theory.
- (ii) The connection between Dirac equation with scalar interaction and supersymmetry is well known[16, 17]. It appears that a connection between supersymmetry and Dirac equation with scalar and pseudo-scalar interactions has been established for the first time in this article. The method outlined in this article for the study of Dirac equation with most general coupling makes an indirect connection with supersymmetry for specific reductions of different interactions. Further investigations along this line may lead to exactly solvable h_{1D} with more general interactions.

4 Dirac Equation in 2 + 1 dimensions

In 2 + 1 dimensions, a non-dissipative non-Dirac-hermitian relativistic quantum system is introduced as follows,

$$\begin{aligned} H_{2D} &= \Sigma^1 \Pi^1 + \Sigma^2 \Pi^2 + \sigma^3 \phi(X^1, X^2) + V(X^1, X^2), \\ \pi^a &= p^a - A^a(x^1, x^2), \quad \Pi^a = P^a - A^a(X^1, X^2), \quad a = 1, 2, \end{aligned} \quad (28)$$

where ϕ is the scalar potential. The interaction V is the time component of a Lorentz vector, while A^1 and A^2 are the space components of the same vector.

The similarity operator ρ_{2D} and the metric η_{2D} are defined as,

$$\rho_{2D} = \exp[-\phi(L^3 + \frac{\sigma^3}{2})], \quad \eta_{2D} := \rho_{2D}^2, \quad (29)$$

which can be obtained from Eqs. (1) and (2) by choosing $n^1 = 0 = n^2, n^3 = 1$. A further choice of $L^3 = 0$ allows to have Rashba-type spin-orbit interaction with purely imaginary coupling in the Hamiltonian H_{2D} . The results of Ref. [12] may be reproduced easily in this special limit. The discussion below is for the general case of $L^3 \neq 0$.

The Hamiltonian H_{2D} is hermitian in the Hilbert space $\mathcal{H}_{\eta_{2D}}$ that is endowed with the metric η_{2D} and can be mapped to a Dirac-hermitian Hamiltonian,

$$\begin{aligned} h_{2D} &= \rho_{2D} H \rho_{2D}^{-1} \\ &= \sigma^1 \pi^1 + \sigma^2 \pi^2 + \sigma^3 \phi(x^1, x^2) + V(x^1, x^2). \end{aligned} \quad (30)$$

An analytically solvable h_{2D} corresponds to an exactly solvable H_{2D} . An example in this regard may be the relativistic Landau problem. In particular, choosing $\phi = 0$, $A^1 = 0$, $A^2 = Bx^1$, $V = Ex^1$ and using the Landau gauge, the Hamiltonian h_{2D} describes a fermion in a crossed uniform electric and magnetic fields with magnitude E and B , respectively. Such a Hamiltonian also appears in the context of graphene and analytical results are known[18].

5 Dirac Equation in 3 + 1 dimensions

A non-Dirac-hermitian relativistic quantum system in 3 + 1 dimensions may be introduced as follows,

$$H_{3D} = \vec{\alpha} \cdot \vec{p} + \beta \Phi(X^1, X^2, X^3) + i\beta\gamma_5 \xi(X^1, X^2, X^3) + V(X^1, X^2, X^3), \quad (31)$$

where the Dirac-representation is used for the matrices $\vec{\alpha}$, β and γ_5 :

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \quad (32)$$

The scalar and the pseudo-scalar potentials are denoted by $\Phi(X^1, X^2, X^3)$ and $\xi(X^1, X^2, X^3)$, respectively. The potential term V can be interpreted as the time component of a Lorentz four-vector. It may be noted that the non-Dirac-hermiticity of H_{3D} is due to the presence of X^1, X^2, X^3 , which are complex functions of their arguments.

Introducing the similarity operator ρ_{3D} and the metric operator η_{3D} ,

$$\rho_{3D} = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \quad \eta_{3D} := \rho_{3D}^2, \quad (33)$$

a non-Dirac-hermitian realization of the Dirac matrices may be obtained through the replacement of $\vec{\alpha}, \beta$ by $\vec{\alpha}_1, \beta_1$:

$$\vec{\alpha}_1 := \begin{pmatrix} 0 & \vec{\Sigma} \\ \vec{\Sigma} & 0 \end{pmatrix}, \quad \beta_1 := \beta. \quad (34)$$

The Γ matrices in this non-Dirac-hermitian realization have the form,

$$\Gamma_0 := \beta, \quad \Gamma_i := \beta\alpha_i, \quad \Gamma_5 := \gamma_5. \quad (35)$$

It may be noted that $\vec{\alpha}_1 \cdot \vec{P} = \vec{\alpha} \cdot \vec{p}$, implying that these two quantities are hermitian in \mathcal{H}_D as well as in $\mathcal{H}_{\eta_{3D}}$. The Hamiltonian can be mapped to a Dirac-hermitian Hamiltonian h_{3D} as,

$$\begin{aligned} h_{3D} &:= \rho_{3D}^{-1} H_{3D} \rho_{3D} \\ &= \vec{\alpha} \cdot \vec{p} + \beta \Phi(x^1, x^2, x^3) + i\beta\gamma_5 \xi(x^1, x^2, x^3) + V(x^1, x^2, x^3). \end{aligned} \quad (36)$$

The Hamiltonian H_{3D} is hermitian in the Hilbert space $\mathcal{H}_{\eta_{3D}}$ with the modified norm $\langle\langle \cdot | \cdot \rangle\rangle_{\eta_{3D}}$. Exactly solvable Dirac-hermitian relativistic quantum systems are very few in $3+1$ dimensions and no attempt has been made in this article to present and/or analyze a non-Dirac-hermitian quantum system with such properties.

6 Summary & Conclusions

A class of non-dissipative non-Dirac-hermitian relativistic quantum systems those are isospectral with known Dirac-hermitian Hamiltonian has been constructed in one, two and three space dimensions. A non-Dirac-hermitian realization of the Dirac matrices and the bosonic operators which are hermitian in a Hilbert space that is endowed with a positive-definite metric enables to construct this class of relativistic quantum systems. The fundamental canonical commutation and anti-commutation relations among different variables remain unchanged for the non-Dirac-hermitian realization. Exact solvability of a subclass of these systems has been shown by establishing and/or employing a connection between supersymmetry and Dirac equation. An indirect connection between Dirac equation with the most general coupling and supersymmetry has been established in the process. It is to be seen whether or not any one or more non-Dirac-hermitian Dirac equations from the large class of models considered in this article have relevance in the realistic physical world.

References

- [1] C. M. Bender, Contemp. Phys. **46**, 277(2005); C.M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243(1998).
- [2] A. Mostafazadeh, Int. J. Geom. Meth. Mod. Phys. **7**, 1191(2010); A. Mostafazadeh, J. Math Phys. **43**, 205(2002); **43**, 2814(2002); **43**, 3944(2002); Nucl. Phys. **B640**, 419(2002).
- [3] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. **213**, 74 (1992).
- [4] P. K. Ghosh, J. Phys. A:Math. Theor. **44**, 215307(2011).

- [5] P. K. Ghosh, J. Phys. A:Math. Theor. **43**, 125203(2010); Int. J. Theor. Phys. **50**, 1143(2011).
- [6] P. K. Ghosh, J. Phys. A38, 7313 (2005); T. Deguchi and P. K. Ghosh, Phys. Rev. **E 80**, 021107(2009); T. Deguchi, P. K. Ghosh and K. Kudo, Phys. Rev. **E 80**, 026213 (2009); T. Deguchi and P. K. Ghosh, J. Phys. **A42**, 475208(2009); P. K. Ghosh and K. S. Gupta, Phys. Lett. **A323**, 29(2004).
- [7] O. Mustafa, J. Phys. **A36**, 5067(2003).
- [8] A. Sinha and P. Roy, Mod. Phys. Lett. **A20**, 2377(2005).
- [9] H. Egrifes and R. Sever, Phys. Lett. **A344**, 117(2005).
- [10] C. -S. Jia and A. de Souza Dutra, J. Phys. **39**, 11877 (2006); C. -S. Jia and A. de Souza Dutra, Ann. Phys. **323**, 566 (2008); V. G. C. S. dos Santos, A. de Souza dutra and M. B. Hott, Phys. Lett. **A373**, 3401 (2009).
- [11] F. Cannata and A. Ventura, Phys. Lett **A372**, 941(2008); J. Phys. **A43**, 075305(2010).
- [12] B. P. Mandal and S. Gupta, Mod. Phys. Lett. **A25**, 1723(2010).
- [13] S. Longhi, Phys. Rev. Lett. **105**, 013903(2010).
- [14] A. Guo et.al., Phys. Rev. Lett. **103**, 093902(2009).
- [15] C. E. Rüter et. al., Nature Phys. **6**, 192(2010).
- [16] F. Cooper, A. Khare and U. Sukhatme, Phys. Rept. **251**, 267(1995).
- [17] Y. Nogami and F. M. Toyama, Phys. Rev. **A47**, 1708(1993); Phys. Rev. **A57**, 93(1998).
- [18] V. Lukose, R. Shankar and G. Baskaran, Phys. Rev. Lett. **98**, 116802(2007).